International Journal of Theoretical Physics, Vol. 31, No. 7, 1992

Topology Lattice as Quantum Logic

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Received November 22, 1991

We discuss the relations between the lattice of topologies for the simplest case of a three-point set and quantum logic. A hypothetical "topologymeter" is considered as a measuring apparatus, and it is shown that it necessarily possesses some quantum features, such as complementarity.

INTRODUCTION

The most striking feature of the lattice of topologies on the set of three (Figure 1) or more points is that this lattice is *nondistributive*. We begin with the principal definitions:

Let X be an arbitrary set. A *topology* on X is a collection τ of subsets of X, called *open*, such that:

T1. $\varnothing, X \in \tau$.
T2. For any $A, B \in \tau$, $A \cap B \in \tau$.
T3. For any collection $A_j \in \tau$, $\bigcup_{j\in J} A_j \in \tau$, where J is an arbitrary index set, $\cup$ and $\cap$ are the usual set union and intersection, respectively.

The topologies on a set X are partially ordered: σ is said to be *weaker* than τ (denoted by $\sigma \leq \tau$) if any set open in σ is open in τ:

$$\sigma \leq \tau \quad \text{iff} \quad \forall A \subset X, \quad A \in \sigma \Rightarrow A \in \tau \tag{0.1}$$

Consider in detail the lattice $\tau(3)$ of all topologies on a set of three points.

We use a brief notation for topologies, listing only the minimal open sets. For example, the notation $\tau = ab$ means $\tau = \{\varnothing, a, b, \{a, b\}, \{a, b, c\}\}$.

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0020-7748/92/0700-1093$06.50/0

Additional examples are

$$\tau = a(ab) = \{\emptyset, a, \{a, b\}, \{a, b, c\}\}$$
$$\tau = a(bc) = \{\emptyset, a, \{b, c\}, \{a, b, c\}\}$$

and so on.

Taking the experimental point of view on logic of Finkelstein (1963) and applying it to the lattice of topologies, we come to the following picture. Imagine that we have a set of apparatus at our disposal. Some of them are used to find the answer to the question "Is σ the topology of the three-point set?" Let us call such an apparatus a topologymeter.

Let some other apparatus work in such a way that it checks the property corresponding to the disjunction of two properties. This means that it gives the answer YES if *at least one* YES is obtained from the devices checking these two properties. We confine ourselves to topologymeters and require the disjunction $\alpha = \beta \vee \gamma$ to be also a topology when both β and γ are topologies.

The conjunction $\beta \wedge \gamma$ is defined as the property measured by the device which gives the answer YES iff the apparatuses measuring β and γ both yield YES.

Now return to the lattice $\tau(3)$ (Figure 1) and consider it as a property lattice. The row of weakest topologies with respect to the partial order (0.1) gives the *atoms* of the lattice. They have the following property: *for any pair of atoms their conjunction is the least element* 0 *of* $\tau(3)$—the undiscrete topology $\{\emptyset, \{a, b, c\}\}$. As will be shown below, it is impossible to define a probability measure on the set of atoms of $\tau(3)$, unlike the property lattice of a classical system. This happens due to *complementarity*. From a quantum

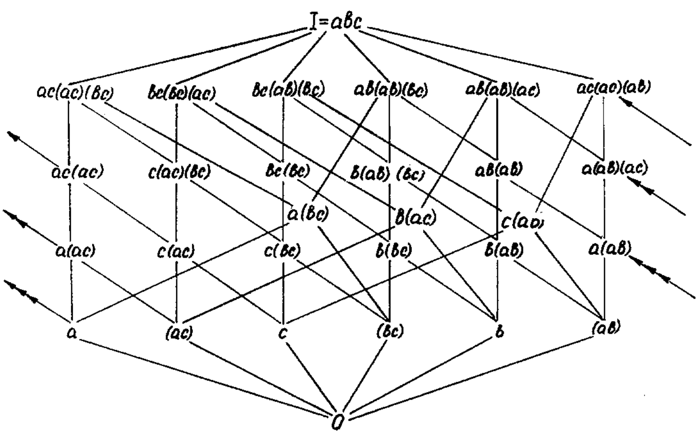

Fig. 1. The lattice $\tau(3)$.

mechanical point of view, this means that it is in principle impossible to use one topologymeter for all topologies in the lower row: several apparatuses must be used for measuring complementary properties. In turn, the complementarity is caused by the nondistributivity of the lattice (see Section 1).

We see the analogous situation in quantum logic. Since quantum logical lattices are nondistributive, we necessarily introduce probability amplitudes instead of probability measures (weights). Thus, the mathematical apparatus describing quantum objects crucially differs from that for classical ones.

We shall try to describe the work of our topologymeter in a way that is similar in many respects to the quantum formalism; call it "quantum topology" for the set of three points. Note that we come to this quantum topology imposing no special postulates such as the quantization of entities which were not "quantum" initially. We come to our picture by considering the topology lattice and comparing it with standard quantum logical lattices for finite quantum mechanical systems (for spin-$\frac{1}{2}$, etc.)

This comparison shows both the similarity and the essential discrepancy between these structures. The object of this paper is to outline them.

1. NONDISTRIBUTIVITY

A lattice L is called distributive if any triple x, y, z of its elements obeys the law of distributivity:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \tag{1.1}$$

When x, y, z are statements of classical logic and $\wedge$ and $\vee$ are the disjunction and conjunction, respectively, the law (1.1) always holds. At the same time, the violation of (1.1) is the main property of the logic of quantum systems.

Now consider Figure 1—the Hasse diagram of the topology lattice for three points.

We emphasize that $\tau(3)$ is nondistributive. In fact, consider the lowest row of atoms of L, the triple a, (ac), (ab). The identity (1, 1)

$$a \wedge [(ac) \vee (ab)] = a \wedge a(ab)(ac) = a$$

$$[a \wedge (ac)] \vee [a \wedge (ab)] = 0 \vee 0 = 0$$

is broken. Analogously, the following triples of topologies are nondistributive: $\{c, (ac), (bc)\}$, $\{b, (ab), (bc)\}$, $\{(ac), a, c\}$, $\{(ab), a, b\}$, $\{(bc), b, c\}$. Thus, as was already mentioned, the usual classical probability cannot be defined on atoms of $\tau(3)$. If we interpret the least element 0 as "false," all the atom-topologies are incompatible with each other. Since none of them is preferred for our topologymeter, it is natural to assume that the "probability" to find any of them is $p = 1/6$. The disjunction $\tau \vee \sigma$ is defined as the least *topology* which is stronger than both τ and σ. Consider the disjunction $a \vee b \vee c = I$, the discrete topology. Since a, b, c are disjoint, the probability $p(a \vee b \vee c)$ must be equal to the sum

$$p(a \vee b \vee c) = p(a) + p(b) + p(c) = \tfrac{1}{6} + \tfrac{1}{6} + \tfrac{1}{6} = \tfrac{1}{2} \tag{1.2}$$

However, $a \vee b \vee c = I$, and $p(I) = 1$, which obviously contradicts (1.2).

In addition, a physicist (Boolean-minded observer) working with our topologymeter will see a strange phenomenon. If the apparatus shows, say, YES for, say, (a), it does not necessarily mean NO for other topologies, in spite of their mutual incompatibility.

Since

$$a = a \wedge ((ac) \vee (ab)) \tag{1.3}$$

the following implications hold:

$$[a \text{ is true}] \Rightarrow [a \wedge ((ac) \vee (ab)) \text{ is true}] \Rightarrow [(ac) \vee (ab) \text{ is true}]$$

The observer is Boolean-minded; thus, for him the latter statement means either (ac) or (ab) *is true* [both cannot be true, since (ac) and (ab) are atoms]. So, he will conclude that with some probability he will see this or that result. The appeal to a Boolean-minded observer measuring topologies corresponds to the von Neumann projection postulate in quantum mechanics giving rise to probabilities.

Due to the same reason (1.3), *complementarity* appears in our topologymeter. Complementarity manifests itself here in incompatibility of, say, (a) and (ac). *Boolean-mindedness* is essential here since in non-Boolean logic $(ac) \vee (ab)$ may be true when both (ac) and (ab) are false.

The Boolean observer will use different apparatuses measuring a, (ab), and (ac) and introduce *time* in such a way that the topology a will be seen at one moment and $(ac) \vee (ab)$ will be seen at some other moment. So, this observer will say that the topology somehow "jumps" instantaneously when the complementary observables are measured.

2. ORTHOGONALITY

The orthogonality in property lattices is very important in quantum logic since it is associated with logical *negation*. The lattice $\tau(3)$ is complemented (Larson and Andima, 1975; Isham, 1989), which means that for any topology $T \in \tau(3)$ there exists a topology τ' such that

$$\tau \vee \tau' = I, \qquad \tau \wedge \tau' = 0$$

However, this τ' is not unique and, moreover, it is impossile to define an orthocomplement on $\tau(3)$. So, the lattice of topologies is complemented, but *not orthocomplemented.* To elucidate this fact, note that the number of elements of any ortholattice must be even, while $\tau(3)$ is of 29 elements. In this sense the whole lattice of topologies cannot be considered as a logic with negation. Nevertheless we can ask the following question: *can a quantum logical lattice be a substructure of a topology lattice?*

Obviously, at least one such substructure does exist. Take the subset of $\tau(3)$ consisting of the lowest row of six proper weakest topologies and the undiscrete topology I (Figure 2). The structure obtained is a subposet, but not a sublattice of $\tau(3)$. It is isomorphic to the lattice M_6 occurring in Grib and Zapatrin (1990, 1991), which corresponds to a property lattice of a spin-$\frac{1}{2}$ system. The properties 1, 3, 5 (2, 4, 6) are "S_x, S_y, $S_z = +1$ (-1)," respectively.

The fact that M_6 is not a sublattice of $\tau(3)$ operationally correspond to the topologymeter measuring nothing besides the atomic topologies and the greatest element I. That means that it gives YES when YES is obtained for at least one of the topologies 1–6. The elements 1–6 can be associated with the topologies (a), (ac), . . . , (b), (ab), respectively. For example, the property 1 means, "We can see the point a as isolated."

The usual quantum formalism can be built on the base of the lattice M_6: the properties 1–6 are associated with projectors in two-dimensional Hilbert space (Grib and Zapatrin, 1990, 1991). One can define the wave function (probability amplitude) instead of the classical probability measure. For our experimental setup we can find the answer to the following question: "if the topology was observed as 1 (preparation of the state), what will be the probability to observe some other topology from this row?" The rule is the usual rule for a spin-$\frac{1}{2}$ system.

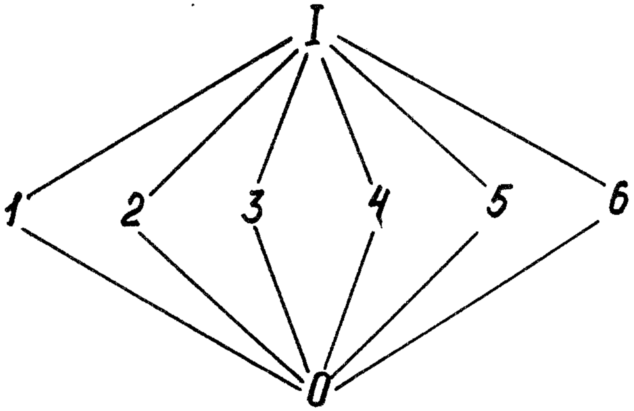

Fig. 2. The lattice M_6.

The non-Boolean lattice M_6 has three Boolean sublattices corresponding to S_y, S_z, S_x (spin projections). These can be the pairs of topologies which we can consider as orthogonal: a, (bc); b, (ac); c, (ab). Then one must have three complementary topologymeters working similar to Stern–Gerlach magnets measuring spin projections. The probability to "find an isolated point" will be calculated as the probability to detect a certain spin projection.

Analogously, instead of the lowest row of $\tau(3)$ (Figure 1), we could take any other one and draw the same lattice M_6 as a subposet of $\tau(3)$. Are there deep reasons for this similarity between our topological lattice and a spin-$\frac{1}{2}$ system? Note that we have no Planck constant for the topological lattice, so that there is nothing microscopic in it.

3. ORTHOMODULARITY

Now let us look for some larger substructures of the whole lattice (Figure 1). For instance, consider a nondistributive subposet in some sense similar to the spin-1 property lattice. We are going to obtain orthomodular lattices, but they are constructed from Boolean blocks (Kalmbach, 1983). However, there is no Boolean subposets of $\tau(3)$ that is larger than 2^3. So, the greatest orthomodular subposet of $\tau(3)$ must contain not more than three proper rows. Denote it by L (see Figure 3).

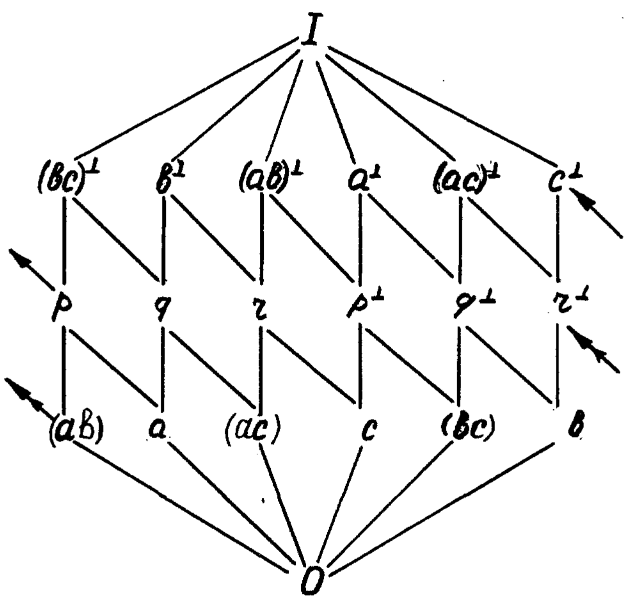

Fig. 3. The lattice L.

Unlike the general lattice (Figure 1), the orthocomplementation can be defined here. So, this lattice can be considered as a logic: the orthocomplementation will play the role of negation.

Then let us try to represent it by projectors in a Hilbert space. Unfortunately, this is impossible since the lattice L is not orthomodular [for details see Kalmbach (1983)]. In fact, consider the pair of elements a, $(bc)^{\perp}$. The element a commutes with $(bc)^{\perp}$, that is,

$$[a \wedge (bc)^{\perp}] \vee [a \wedge (bc)] = a$$

while $(bc)^{\perp}$ does not commute with a,

$$[(bc)^{\perp} \wedge a] \vee [(bc)^{\perp} \wedge a^{\perp}] = a \neq (bc)^{\perp}$$

This means that the lattice L cannot be represented by projectors in a Hilbert space: it is not orthomodular. Consequently, the object whose property lattice is L cannot be described as a quantum system.

4. ANALYSIS OF EPR EXPERIMENT WITH TWO SPIN-$\frac{1}{2}$ PARTICLES

Here we discuss the possible application of the lattice of topologies for a three-point set to the Einstein–Podolsky–Rosen experiment with two spin-$\frac{1}{2}$ particles.

Let the particles be prepared in singlet state with $s = 0$. This preparation is made at some point a and corresponds to space-time event "a." Even when these two particles are emitted, they are still described by an antisymmetrized wave function

$$\psi(x_b, x_c) = \frac{1}{\sqrt{2}} [\psi_1(x_b)\psi_2(x_c) - \psi_1(x_c)\psi_2(x_b)]$$

This wave function is associated with the nonlocal event (bc). We use (bc) in order to stress that "b" and "c" do not exist as separate entities. This wave function is an eigenstate of the nonlocal permutation operator $\hat{P}_{12}$ which does not commute with operators of local coordinates for both particles 1 and 2.

So, our hypothesis is to put the arguments, of a many-particle wave function in multidimensional configuration space into correspondence with some topology: here it is (bc). As is well known (D'Espagnat, 1976), the event 1 in Minkowski space-time appears due to measurement. One of the main lines of reasoning within EPR situations showing why Bell's inequalities (D'Espagnet, 1976) are broken is the following. If the observer 1 has measured spin projection $S_z^{(1)} = +\frac{1}{2}$ for particle 1 at point "b," then he will say with probability 1 that the other observer will see $S_z^{(2)} = -\frac{1}{2}$ at the other point "c." But can observer 1 say that *before* the observation performed by 2 there is an "event" at point "c?" Surely the answer is negative. The observer 2 could choose to measure not S_z, but S_x and then at c another event would occur. More evident proof of the nonexistence of the event at c before observation is based on the relativistic definition of simultaneous events.

A reference frame always exists where some event prior to b, if it exists, is simultaneous to event c. Therefore the properties described by noncommuting operators exist before their observation (S_z is in no sense better than S_x as some objective reality). But this assumption implies Bell's inequalities, which, as we know, are broken for quantum particles. So the question is: "What does the observer do as *event creator* in the EPR experiment?" Our answer is: "*He chooses the topology!*" He takes out b from (bc) and c from (bc). So Minkowski space-time events appear due to three measurements of a, b, and c. The event a corresponds to the emission of the prepared pair of particles. The observer 1 chooses $b\,(bc)$ so that b is isolated. The observer 2 chooses $c\,(bc)$. Both observers create $bc\,(bc)$, which together with the observer who prepared the wave function at point a constitute the whole I.

We emphasize the importance of the third point a in the EPR experiment. The lattice $\tau(2)$ of topologies on a two-point set is distributive, unlike $\tau(3)$. That is why (bc) here means $(bc)\,(abc)$, b means $b\,(abc)$, and so on. It is impossible in EPR experiments to have only two points b and c in space-time. This is caused by the noncommutativity of $\hat{P}_{12}$ with local observables. $\hat{P}_{12}$ corresponds to the preparation of the two-particle state at a moment of time other than the measurement of local observables. So in an EPR experiment the third point a must always occur, separated by some time interval from (bc).

5. SUMMARY

Let us look at the lattice $\tau(3)$ of topologies on a three-point set as a property lattice. Then, due to its nondistributivity, some kind of complementarity immediately arises as in quantum theory. That makes it impossible to use traditional probability calculus for topologies.

Now consider the whole lattice (Figure 1) as some logic. We see that the lack of orthogonality makes it impossible to define negation in this logic. In order to introduce the negation, one could take some substructure of the whole lattice. It comes out that such a substructure can only be composed of one row of elements of $\tau(3)$ together with its greatest (I) and least (0) elements. The obtained substructure can be associated with the well-known spin-$\frac{1}{2}$ quantum mechanical system for which the spin projections S_x, S_y, S_z are measured. The nondistributivity of the lattice corresponds to the noncommutativity of the operators S_x, S_y, S_z.

For more complex substructures, we could define the orthogonality considered as the negation in an appropriate logic. However, this logic is not a logic of a quantum system since it is not orthomodular.

So the lattice $\tau(3)$ is an interesting example of something more general than the usual quantum system. It has substructures corresponding to quantum systems which are realizations of a quantum formalism beyond microphysics. Generally the collection of topologies can be thought of as some new physical object. Its complete description must be realized by a formalism more general than the quantum mechanical one. Some hints along these lines are given in Zapatrin (1989, 1992).

ACKNOWLEDGMENT

One of the authors (A.A.G.) is greatly indebted to C. Isham for fruitful discussions.

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